Concavity of Certain Maps on Positive Definite Matrices and Applications to Hadamard Products*

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ABSTRACT

If f is a positive function on $(0,\infty)$ which is monotone of order n for every n in the sense of Löwner and if Φ_1 and Φ_2 are concave maps among positive definite matrices, then the following map involving tensor products:

$$(A,B)\mapsto f\left[\Phi_1(A)^{-1}\otimes\Phi_2(B)\right]\cdot(\Phi_1(A)\otimes I)$$

is proved to be concave. If Φ_1 is affine, it is proved without use of positivity that the map

$$(A,B)\mapsto f\left[\Phi_1(A)\otimes\Phi_2(B)^{-1}\right]\cdot(\Phi_1(A)\otimes I)$$

is convex. These yield the concavity of the map

$$(A,B)\mapsto A^{1-p}\otimes B^{p}\qquad(0<p\leqslant 1)$$

(Lieb's theorem) and the convexity of the map

$$(A,B)\mapsto A^{1+p}\otimes B^{-p}\qquad(0<p\leqslant 1),$$

as well as the convexity of the map

$$(A,B)\mapsto(A\cdot\log[A])\otimes I-A\otimes\log[B].$$

These concavity and convexity theorems are then applied to obtain unusual estimates, from above and below, for Hadamard products of positive definite matrices.

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INTRODUCTION

It has long been known that in the case $0 \leqslant p \leqslant 1$ the map $A \mapsto A^p$ is concave on the space of positive definite matrices. Here concavity and convexity are understood in terms of the natural order structure associated with the notion of positive semidefiniteness. But only recently has Lieb [11] succeeded in proving the concavity of the map $(A,B) \mapsto A^{1-p} \otimes B^p$, the tensor product, in the case $0 < p \leqslant 1$. The purpose of this paper is to establish concavity and convexity theorems of this type in full generality, and to apply them to obtain unusual estimates for Hadamard products of positive definite matrices.

Our main theorems, proved in Sec. 4, will show that if $f(\lambda)$ is a real valued continuous function on $(0, \infty)$, which is monotone of order n for every n in the sense of Löwner (*operator-monotone* in our terminology), and if Φ_1 and Φ_2 are concave maps, then the map

$$(A,B) \mapsto f\left[\Phi_1(A)^{-1} \otimes \Phi_2(B)\right] \cdot \left(\Phi_1(A) \otimes I\right)$$

is concave, A and B being positive definite. If Φ_1 is affine in the above, then the map

$$(A,B) \mapsto f\left[\Phi_1(A) \otimes \Phi_2(B)^{-1}\right] \cdot \left(\Phi_1(A) \otimes I\right)$$

is convex. These theorems yield as corollaries, besides Lieb's theorem, convexity of the following maps:

$$(A,B) \mapsto A^{p+1} \otimes B^{-p} \qquad (0 \leqslant p \leqslant 1)$$

and

$$(A,B) \mapsto (A \cdot \log[A]) \otimes I - A \otimes \log[B].$$

In the proof of the main theorems binary operations for a pair of positive definite matrices, called *harmonic mean* and *geometric mean*, play an important role. In Sec. 1 we present basic concavity theorems for these operations. In Sec. 2 filtration of these operations through a positive linear map is discussed. Incidentally for a positive linear map Φ the concavity of the map $A \mapsto \Phi(A^{-1})^{-1}$ is proved, A being positive definite. Section 3 is a survey of Löwner and Davis's theory on operator-monotone functions, arranged to our purpose.

Applicability of the main results comes from the fact that the Hadamard product $A*B$, the entrywise product of A and B, is obtained by filtering the tensor product $A\otimes B$ through a positive linear map. Among various inequalities obtained for Hadamard products are the following. (Here the A's and B's are positive definite matrices.) *Hölder's inequalities* are

$$\sum_{i=1}^{k}(A_i*B_i)\leqslant\left\{\sum_{i=1}^{k}A_i^p\right\}^{1/p}*\left\{\sum_{i=1}^{k}B_i^q\right\}^{1/q}\qquad\text{if}\quad p,q\geqslant 1\ \text{ and }\ \frac{1}{p}+\frac{1}{q}=1$$

and

$$\sum_{i=1}^{k}(A_i*B_i)\geqslant\left\{\sum_{i=1}^{k}A_i^{-p}\right\}^{-1/p}*\left\{\sum_{i=1}^{k}B_i^q\right\}^{1/q}$$

$$\text{if}\quad p\geqslant 1\geqslant q>0\ \text{ and }\ \frac{1}{q}-\frac{1}{p}=1.$$

These yield as corollaries

$$\tfrac{1}{2}(A^2+B^2)^{1/2}*(A^2+B^2)^{1/2}\geqslant A*B\geqslant\tfrac{1}{2}(A^{-1}+B^{-1})^{-1}*(A^{1/2}+B^{1/2})^2.$$

The right hand inequality is improved further:

$$A*B\geqslant(A^{-1}+B^{-1})^{-1}*(A+B).$$

The diagonal matrix formed from A is obtained as the Hadamard product of A and the identity matrix I. Then Hölder's inequalities give estimates by diagonal matrices:

$$A*B\leqslant(A^p*I)^{1/p}(B^q*I)^{1/q}\qquad\text{if}\quad p,q\geqslant 1\ \text{ and }\ \frac{1}{p}+\frac{1}{q}=1$$

and

$$A*B\geqslant(A^{-p}*I)^{-1/p}(B^q*I)^{1/q}\qquad\text{if}\quad p\geqslant 1\geqslant q>0\ \text{ and }\ \frac{1}{q}-\frac{1}{p}=1.$$

Another remarkable inequality is

$$\log[A*B]\geqslant\{\log[A]+\log[B]\}*I,$$

which implies Fiedler's theorem [10]: $A * A^{-1} \geqslant I$. Further, Styan's inequality [15, p. 236] is derived in a very simple way.

We shall use a halmos ■ to denote "end of proof."

CONCAVITY THEOREMS

1. *Basic operations*

The space $\mathbf{H}_n$ of n-square Hermitian matrices is provided with the natural order structure; the relation $A \geqslant B$ means that $A - B$ is positive semidefinite. The open cone of positive definite matrices will be denoted by $\mathbf{H}_n^+$. We use, without further reference, the known fact (see [6, p. 198]) that for positive definite A, B the relation $A \geqslant B$ implies $A^{-1} \leqslant B^{-1}$. Given a positive definite A, its positive definite square root is denoted by $A^{1/2}$. The following fact is known as the Löwner-Heinz theorem (see e.g. [6, p. 199]); for positive definite A, B the relation $A \geqslant B$ implies $A^{1/2} \geqslant B^{1/2}$.

A map Φ from the product $\mathbf{H}_{n_1}^+ \times \cdots \times \mathbf{H}_{n_k}^+$ to $\mathbf{H}_m$ is said to be *convex* if

$$\Phi(\lambda A_1 + (1-\lambda)B_1, \dots, \lambda A_k + (1-\lambda)B_k) \leqslant \lambda \Phi(A_1, \dots, A_k) + (1-\lambda)\Phi(B_1, \dots, B_k)$$

for any A_i, B_i in $\mathbf{H}_{n_i}^+$ $(i = 1, \dots, k)$ and $0 < \lambda < 1$. Φ is *concave* by definition if the map $(A_1, \dots, A_k) \mapsto -\Phi(A_1, \dots, A_k)$ is convex.

In this section we introduce two binary operations, called harmonic mean and geometric mean, for each pair of positive definite matrices, which play a fundamental role in the subsequent sections. Basic properties of the harmonic mean and geometric mean have been established by Anderson and Duffin [2] and Pusz and Woronowicz [14]. But our approach is somewhat different.

We start with a folklore lemma (see e.g. [1]).

LEMMA 1. *For positive definite A, B and any matrix C, the block matrix $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$, C^* being the Hermitian transpose of C, is positive semidefinite if and only if $B \geqslant C^* A^{-1} C$.*

The following theorem is a variant of a result of Anderson [1, p. 521].

THEOREM 1. *Given positive definite A, B, the matrix $BA^{-1}B$ is the minimum of all Hermitian C for which $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is positive semidefinite. The map $(A, B) \mapsto BA^{-1}B$ is convex on $\mathbf{H}_n^+ \times \mathbf{H}_n^+$. In particular, the maps $A \mapsto A^{-1}$ and $A \mapsto A^2$ are convex on $\mathbf{H}_n^+$.*

Proof. The first part is merely a restatement of Lemma *1*. Let A_i, B_i be positive definite $(i=1,2)$ and $0<\lambda<1$. Since by the first part

$$0 \leqslant \lambda \begin{bmatrix} A_1 & B_1 \\ B_1 & B_1A_1^{-1}B_1 \end{bmatrix} + (1-\lambda)\begin{bmatrix} A_2 & B_2 \\ B_2 & B_2A_2^{-1}B_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda A_1 + (1-\lambda)A_2 & \lambda B_1 + (1-\lambda)B_2 \\ \lambda B_1 + (1-\lambda)B_2 & \lambda B_1A_1^{-1}B_1 + (1-\lambda)B_2A_2^{-1}B_2 \end{bmatrix},$$

the minimum property implies convexity:

$$\lambda B_1A_1^{-1}B_1 + (1-\lambda)B_2A_2^{-1}B_2$$

$$\geqslant \{\lambda B_1 + (1-\lambda)B_2\}\{\lambda A_1 + (1-\lambda)A_2\}^{-1}\{\lambda B_1 + (1-\lambda)B_2\}.$$

∎

Let us define the *harmonic mean* $A\,!\,B$ of positive definite A and B by

$$A\,!\,B \equiv \left\{\tfrac{1}{2}(A^{-1}+B^{-1})\right\}^{-1}.$$

Remark that $\frac{1}{2}(A\,!\,B)$ is just the *parallel sum* of A and B introduced by Anderson and Duffin [2].

COROLLARY 1.1 (Anderson and Duffin). *The map $(A,B)\mapsto A\,!\,B$ is concave.*

Proof. The assertion follows from Theorem 1 through the identity

$$A\,!\,B = 2\left\{B - B(A+B)^{-1}B\right\}.$$

∎

COROLLARY 1.2. *Let Φ and Ψ be maps from $\mathbf{H}_n^+$ to $\mathbf{H}_m^+$. If Φ is concave and if Ψ is affine, then the map $A\mapsto\Psi(A)\Phi(A)^{-1}\Psi(A)$ is convex.*

Proof. By Theorem 1, for $0<\lambda<1$,

$$\{\lambda\Psi(A)+(1-\lambda)\Psi(B)\}\{\lambda\Phi(A)+(1-\lambda)\Phi(B)\}^{-1}\{\lambda\Psi(A)+(1-\lambda)\Psi(B)\}$$

$$\leqslant \lambda\Psi(A)\Phi(A)^{-1}\Psi(A)+(1-\lambda)\Psi(B)\Phi(B)^{-1}\Psi(B).$$

But the affine property of Ψ implies

$$\lambda\Psi(A)+(1-\lambda)\Psi(B)=\Psi(\lambda A+(1-\lambda)B),$$

while concavity of Φ implies

$$\{\lambda\Phi(A)+(1-\lambda)\Phi(B)\}^{-1}\geqslant\Phi(\lambda A+(1-\lambda)B)^{-1}. \qquad \blacksquare$$

REMARK. (1) The map $(A,B)\mapsto A^{-1/2}B^2A^{-1/2}$ is not always convex.

(2) The harmonic mean $A\,!\,B$ can be characterized as the maximum of all Hermitian C for which

$$2\begin{bmatrix} A & 0 \\ 0 & B\end{bmatrix}\geqslant\begin{bmatrix} C & C \\ C & C\end{bmatrix}.$$

(3) If Φ is convex, the map $A\mapsto\Phi(A)^{-1}$ is not always concave.

When restricted to the case of matrices, Pusz and Woronowicz [14] proved that for each pair of positive definite A,B there exists the maximum, called the *geometric mean* of A and B, of all Hermitian C for which $\begin{bmatrix} A & C \\ C & B\end{bmatrix}$ is positive semidefinite. The following theorem gives an explicit form.

THEOREM 2 (Pusz and Woronowicz). *For positive definite A,B, the geometric mean is given by*

$$A\,\#\,B\equiv A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

The map $(A,B)\mapsto A\,\#\,B$ is concave.

Proof. Since

$$(A\,\#\,B)A^{-1}(A\,\#\,B)=A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}=B,$$

the matrix $\begin{bmatrix} A & A\,\#\,B \\ A\,\#\,B & B\end{bmatrix}$ is positive semidefinite by Lemma 1. If conversely $\begin{bmatrix} A & C \\ C & B\end{bmatrix}$ is positive semidefinite, then again by Lemma 1 we have $B\geqslant CA^{-1}C$; hence

$$(A^{-1/2}CA^{-1/2})^2\leqslant A^{-1/2}BA^{-1/2}.$$

Then the Löwner-Heinz theorem implies $A^{-1/2}CA^{-1/2} \leqslant (A^{-1/2}BA^{-1/2})^{1/2}$, which leads to $C \leqslant A \# B$. Thus the maximum property of $A \# B$ is established. The concavity of the map $(A,B) \mapsto A \# B$ follows from this maximum property as the convexity of the map $(A,B) \mapsto BA^{-1}B$ does from the minimum property of $BA^{-1}B$. ∎

REMARK. If A commutes with B, then $A \# B = (AB)^{1/2}$. In particular, $A^p \# A^q = A^{(p+q)/2}$ for all $-\infty < p, q < \infty$.

COROLLARY 2.1. *The harmonic mean and geometric mean enjoy the following properties:*

(i) $A ! B = B ! A$ *and* $A \# B = B \# A$.

(ii) $C^*(A ! B)C = (C^*AC) ! (C^*BC)$ *and* $C^*(A \# B)C = (C^*AC) \# (C^*BC)$ *for invertible (non-Hermitian)* C.

(iii) $(A ! B)^{-1} = \frac{1}{2}(A^{-1} + B^{-1})$ *and* $(A \# B)^{-1} = A^{-1} \# B^{-1}$.

(iv) $\frac{1}{2}(A+B) \geqslant A \# B \geqslant A ! B$.

(v) $\{\frac{1}{2}(A+B)\} \# (A ! B) = A \# B$.

Proof. (i): $A ! B = B ! A$ follows from definition of the harmonic mean, while $A \# B = B \# A$ follows from the maximum property. (ii) is proved analogously. (iii) is obvious. By using (ii), the proof of (iv) is reduced to that of the inequalities

$$\tfrac{1}{2}(I+D) \geqslant D^{1/2} \geqslant \left\{\tfrac{1}{2}(I+D^{-1})\right\}^{-1} \qquad \text{with} \quad D = A^{-1/2}BA^{-1/2}.$$

These inequalities follow from arithmetic-geometric-harmonic inequalities via functional calculus for the positive definite matrix D. Finally, by (ii)

$$\begin{aligned}\left\{\tfrac{1}{2}(A+B)\right\} \# (A ! B) &= A^{1/2}\left\{\left\{\tfrac{1}{2}(I+D)\right\} \# (I ! D)\right\}A^{1/2} \\ &= A^{1/2}\left\{\left\{\tfrac{1}{2}(I+D)\right\}^{1/2}\left\{\tfrac{1}{2}(I+D^{-1})\right\}^{-1/2}\right\}A^{1/2} \\ &= A^{1/2}D^{1/2}A^{1/2} = A \# B,\end{aligned}$$

which proves (v). ∎

COROLLARY 2.2. *If* Φ_i *is a concave map from* $\mathbf{H}_{n_i}^+$ *to* $\mathbf{H}_m^+$ $(i=1,2)$, *then the maps* $(A_1, A_2) \mapsto \Phi_1(A_1) ! \Phi_2(A_2)$ *and* $(A_1, A_2) \mapsto \Phi_1(A_1) \# \Phi_2(A_2)$ *are concave.*

Proof. Since the map $(X,Y)\mapsto X\,!\,Y$ is concave by Theorem 1 and positively homogeneous by Corollary 2.1(ii), it is *monotone* in the sense that $X_1\,!\,Y_1 \geqslant X_2\,!\,Y_2$ whenever $X_1 \geqslant X_2$ and $Y_1 \geqslant Y_2$. Now to prove the concavity of $\Phi_1\,!\,\Phi_2$, take A_1, B_1 in $\mathbf{H}_{n_1}^+$ and A_2, B_2 in $\mathbf{H}_{n_2}^+$ and $0<\lambda<1$. Concavity in assumption implies

$$\Phi_i(\lambda A_i + (1-\lambda)B_i) \geqslant \lambda\Phi_i(A_i) + (1-\lambda)\Phi_i(B_i) \qquad (i=1,2).$$

Then the monotonicity just pointed out, with the concavity of the operation !, yields

$$\Phi_1(\lambda A_1 + (1-\lambda)B_1)\,!\,\Phi_2(\lambda A_2 + (1-\lambda)B_2)$$

$$\geqslant \lambda\{\Phi_1(A_1)\,!\,\Phi_2(A_2)\} + (1-\lambda)\{\Phi_1(B_1)\,!\,\Phi_2(B_2)\}.$$

The assertion about $\Phi_1 \# \Phi_2$ is proved analogously. ∎

2. *Positive linear maps*

A linear map from $\mathbf{H}_n$ to $\mathbf{H}_m$ is said to be *positive* if it transforms $\mathbf{H}_n^+$ into $\mathbf{H}_m^+$. The restriction of a positive linear map to $\mathbf{H}_n^+$ is affine. A positive linear map is said to be *normalized* if it maps I_n, the identity matrix, to I_m. Remark that a positive linear map Φ is *monotone* in the sense that $A \leqslant B$ implies $\Phi(A) \leqslant \Phi(B)$.

In this section we study how geometric and harmonic means are modified when filtered through a positive linear map.

The following is a basic property of a positive linear map in connection with algebraic operations.

LEMMA 2 (Kadison (see e.g. [6])). *If Φ is a normalized positive linear map, then for any positive definite A*

$$\Phi(A)^2 \leqslant \Phi(A^2) \quad \textit{and} \quad \Phi(A)^{-1} \leqslant \Phi(A^{-1}).$$

THEOREM 3. *If Φ is a positive linear map, then for any positive definite A, B*

(i) $\Phi(A \# B) \leqslant \Phi(A) \# \Phi(B)$, *and*
(ii) $\Phi(A\,!\,B) \leqslant \Phi(A)\,!\,\Phi(B)$.

Proof. Let A and B be positive definite. Consider the map Ψ defined by

$$\Psi(X) \equiv \Phi(B)^{-1/2}\Phi(B^{1/2}XB^{1/2})\Phi(B)^{-1/2}.$$

Since Ψ is a positive linear map as Φ and is normalized, by Lemma 2 and the Löwner-Heinz theorem $\Psi(X^{1/2}) \leqslant \Psi(X)^{1/2}$ for all positive definite X. Therefore

$$\begin{aligned}\Psi\big((B^{-1/2}AB^{-1/2})^{1/2}\big) &\leqslant \Psi(B^{-1/2}AB^{-1/2})^{1/2}\\ &= \{\Phi(B)^{-1/2}\Phi(A)\Phi(B)^{-1/2}\}^{1/2},\end{aligned}$$

which implies (i):

$$\begin{aligned}\Phi(B \# A) &= \Phi(B)^{1/2}\Psi\big((B^{-1/2}AB^{-1/2})^{1/2}\big)\Phi(B)^{1/2}\\ &\leqslant \Phi(B)^{1/2}\{\Phi(B)^{-1/2}\Phi(A)\Phi(B)^{-1/2}\}^{1/2}\Phi(B)^{1/2}\\ &= \Phi(B) \# \Phi(A).\end{aligned}$$

(ii) follows analogously from another inequality in Lemma 2. ∎

The assertion (ii) was proved, under more restrictive conditions on Φ, by Anderson and Trapp [3].

The following result unifies inequalities in Lemma 2 into a single form without the presence of normality.

COROLLARY 3.1. *If Φ is a positive linear map, then for any positive definite A, B*

$$\Phi(B)\Phi(A)^{-1}\Phi(B) \leqslant \Phi(BA^{-1}B).$$

Proof. Let Ψ be the normalized positive linear map in the proof of Theorem 3. Then by Lemma 2

$$\begin{aligned}\Phi(B)^{1/2}\Phi(A)^{-1}\Phi(B)^{1/2} &= \Psi(B^{-1/2}AB^{-1/2})^{-1}\\ &\leqslant \Psi\big((B^{-1/2}AB^{-1/2})^{-1}\big) = \Phi(B)^{-1/2}\Phi(BA^{-1}B)\Phi(B)^{-1/2}. \quad ∎\end{aligned}$$

COROLLARY 3.2. *If Φ is a positive linear map from $\mathbf{H}_n$ to $\mathbf{H}_m$, then the map $A \mapsto \Phi(A^{-1})^{-1}$ is concave on $\mathbf{H}_n^+$.*

Proof. Since this new map is continuous, it suffices to prove "midpoint" concavity. But Theorem 3 implies that for positive definite A, B

$$\Phi\Big(\big\{\tfrac{1}{2}(A+B)\big\}^{-1}\Big)^{-1} = \Phi(A^{-1} ! B^{-1})^{-1}$$

$$\geqslant \{\Phi(A^{-1}) ! \Phi(B^{-1})\}^{-1} = \tfrac{1}{2}\big\{\Phi(A^{-1})^{-1} + \Phi(B^{-1})^{-1}\big\}. \qquad \blacksquare$$

REMARK. When combined with Lemma 1, Corollary 3.1 shows that if the matrix $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is positive semidefinite with positive definite B, then $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(B) & \Phi(C) \end{bmatrix}$ is positive semidefinite for any positive linear map Φ. This assertion is not valid without positive semidefiniteness of B (see [4]).

3. *Operator-monotone functions*

In this section we study the concavity property of certain maps induced by functional calculus for Hermitian matrices and filtration of those maps through positive linear maps.

The spectral theorem (see e.g. [13, p. 67]) assures that for an n-square Hermitian matrix A there are a unitary matrix U and real numbers (*eigenvalues* of A) $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ such that

$$A = U^* \operatorname{diag}(\lambda_i)\, U,$$

where $\operatorname{diag}(\lambda_i)$ denotes the diagonal matrix with diagonal entries (λ_i). If f is a real valued continuous function defined on an interval containing all λ_i, the Hermitian matrix $f[A]$ is defined by

$$f[A] = U^* \operatorname{diag}(f(\lambda_i))\, U.$$

For any such functions f, g, the matrices $f[A]$ and $g[A]$ commute and

$$(f+g)[A] = f[A] + g[A] \quad \text{and} \quad (fg)[A] = f[A] \cdot g[A].$$

Further, if $f(\lambda) \leqslant g(\lambda)$ for all λ, then

$$f[A] \leqslant g[A].$$

A real valued continuous function f on $(0, \infty)$ is said to be *operator-monotone* if for every n and all n-square positive definite matrices A, B the relation $A \leqslant B$ implies $f[A] \leqslant f[B]$. Thus that f is operator-monotone means that it is a *monotone-matrix function* of order n for every n in the sense of Löwner (see [6], [8]).

An operator-monotone function is nondecreasing in the usual sense, but the converse is not true. A part of the deep theory of Löwner is summarized in the following lemma. A complete account of Löwner's theory can be found in the book [8] as well as [6].

LEMMA 3 (Löwner). *The following statements for a real valued continuous function f on $(0, \infty)$ are equivalent:*

(i) *f is operator-monotone.*

(ii) *f admits an analytic continuation to the whole domain* $\operatorname{Im} z \neq 0$ *in such a way that* $\operatorname{Im} f(z) \cdot \operatorname{Im} z \geqslant 0$.

(iii) *f admits an integral representation*

$$f(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{0} (1 + \lambda t)(t - \lambda)^{-1} d\mu(t) \qquad \text{for} \quad \lambda > 0,$$

where α is real, β is non-negative and μ is a finite positive measure on $(-\infty, 0]$.

If f is operator-monotone, the integral representation in Lemma 3 is valid even for complex λ with $\operatorname{Im}\lambda \neq 0$. If, in addition, f is positive and nonconstant, its analytic continuation maps the open upper (or the lower) half plane into itself by Lemma 3, so that the function $g(z) \equiv f(z^{-1})^{-1}$ has the same property, and induces an operator-monotone function on $(0, \infty)$ again by Lemma 3.

The most important examples of operator-monotone functions are $\log\lambda$ and λ^p $(0 < p \leqslant 1)$:

$$\log\lambda = \int_{-\infty}^{0} (1 + \lambda t)(t - \lambda)^{-1} (1 + t^2)^{-1} dt,$$

and for $0 < p \leqslant 1$

$$\lambda^p = \cos(2^{-1} p\pi) + \pi^{-1} \sin(p\pi) \int_{-\infty}^{0} (1 + \lambda t)(t - \lambda)^{-1} |t|^p (1 + t^2)^{-1} dt$$

$$= \pi^{-1} \sin(p\pi) \int_{0}^{\infty} \lambda(\lambda + t)^{-1} t^{p-1} dt.$$

THEOREM 4. *If a function f is operator-monotone on $(0,\infty)$, then the map $A \mapsto f[A]$ is concave on $\mathbf{H}_n^+$, while the map $A \mapsto A \cdot f[A]$ is convex. If Φ is a normalized positive linear map on $\mathbf{H}_n$, then for any positive definite A*

$$f[\Phi(A)] \geqslant \Phi(f[A])$$

and

$$\Phi(A) \cdot f[\Phi(A)] \leqslant \Phi(Af[A]).$$

Proof. Let f admit an integral representation in Lemma 3:

$$f(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{0} (1+\lambda t)(t-\lambda)^{-1} d\mu(t).$$

Then for any positive definite A

$$f[A] = \alpha I_n + \beta A + \int_{-\infty}^{0} (I_n + tA)(tI_n - A)^{-1} d\mu(t).$$

Obviously the map $A \mapsto \alpha I_n + \beta A$ is concave. Since further for $t \leqslant 0$ the map

$$A \mapsto (I_n + tA)(tI_n - A)^{-1} = -tI_n - (1+t^2)(A - tI_n)^{-1}$$

is concave by Theorem 1, the map $A \mapsto f[A]$ is concave too. Analogously

$$A \cdot f[A] = \alpha A + \beta A^2 + \int_{-\infty}^{0} A(I_n + tA)(tI_n - A)^{-1} d\mu(t).$$

Since $\beta \geqslant 0$ by Lemma 3, the map $A \mapsto \alpha A + \beta A^2$ is convex by Theorem 1, while for $t \leqslant 0$ the map

$$\begin{aligned} A &\mapsto A(I_n + tA)(tI_n - A)^{-1} \\ &= -tA - \tfrac{1}{2}|t|^{-1}(1+t^2)\{A \,!\, (|t| I_n)\} \end{aligned}$$

is convex by Corollary 1.1. Therefore the map $A \mapsto A \cdot f[A]$ is convex.

The assertions on filtration by a normalized positive linear map were proved by Davis [5, 6]. Indeed, these follow immediately from Lemma 2 and the above integral representation. ∎

COROLLARY 4.1. (*i*) *The map* $A \mapsto A^p$ *on* $\mathbf{H}_n^+$ *is concave if* $0<p\leqslant 1$, *and is convex if* $1\leqslant p\leqslant 2$ *or* $-1\leqslant p<0$.

(ii) *The map* $A\mapsto \log[A]$ *is concave, while the map* $A\mapsto A\cdot\log[A]$ *is convex.*

Proof. Since λ^p $(0<p\leqslant 1)$ and $\log\lambda$ are operator-monotone, all the assertions follow from Theorem 4, except the convexity of $A\mapsto A^p$ for $-1\leqslant p<0$, which is proved by additional use of Corollary 1.2. ■

COROLLARY 4.2. *Let* Φ *be a normalized positive linear map from* $\mathbf{H}_n$ *to* $\mathbf{H}_m$ *and* A *positive definite. Then:*

(i) $\Phi(A)\leqslant\Phi(A^p)^{1/p}$ *if* $1\leqslant p<\infty$.
(ii) $\Phi(A)\geqslant\Phi(A^p)^{1/p}$ *if* $\frac{1}{2}\leqslant p\leqslant 1$.
(iii) $\Phi(A)\geqslant\Phi(A^{-p})^{-1/p}$ *if* $1\leqslant p<\infty$.
(iv) $\Phi(\log[A])\leqslant\log[\Phi(A)]$.
(v) $\Phi(A\cdot\log[A])\geqslant\Phi(A)\cdot\log[\Phi(A)]$.

Proof. Since $\lambda^{1/p}$ is operator-monotone for $1\leqslant p<\infty$, (i) follows from Theorem 4. Analogously (ii) results from the operator-monotonicity of $\lambda^{1/p-1}$ for $\frac{1}{2}\leqslant p\leqslant 1$. Since $\Phi(A^{-1})\leqslant\Phi(A^{-p})^{1/p}$ for $1\leqslant p<\infty$ by (i), (iii) follows from Lemma 2. Finally (iv) and (v) follow from Theorem 4 because of the operator-monotonicity of $\log\lambda$. ■

COROLLARY 4.3. *Let* f *be a positive operator-monotone function on* $(0,\infty)$, *and* $0<p\leqslant 1$.

(i) *Both the maps* $A\mapsto f[A^p]^{1/p}$ *and* $A\mapsto f[A^{-p}]^{-1/p}$ *are concave on* $\mathbf{H}_n^+$.
(ii) *Both the maps* $A\mapsto f[A^p]^{-1/p}$ *and* $A\mapsto f[A^{-p}]^{1/p}$ *are convex.*

Proof. (ii) will follow from (i) by Corollary 1.2. In view of Theorem 4, to see (i), it suffices to show that both the functions $h(\lambda)\equiv f(\lambda^p)^{1/p}$ and $g(\lambda)\equiv h(\lambda^{-1})^{-1}$ are operator-monotone on $(0,\infty)$. The operator-monotonicity of g follows from that of h, as pointed out after Lemma 3. Let f admit an integral representation as in Lemma 3:

$$f(\lambda)=\alpha+\beta\lambda+\int_{-\infty}^{0}(1+\lambda t)(t-\lambda)^{-1}d\mu(t).$$

Since f is positive and nondecreasing,

$$0\leqslant\gamma\equiv\lim_{\lambda\to 0+}f(\lambda)=\alpha+\int_{-\infty}^{0}t^{-1}d\mu(t);$$

hence

$$f(\lambda)=\gamma+\beta\lambda+\int_{-\infty}^{0}\{(1+\lambda t)(t-\lambda)^{-1}-t^{-1}\}\,d\mu(t)$$
$$=\gamma+\beta\lambda+\int_{0}^{\infty}\lambda(\lambda+t)^{-1}\,d\nu(t),$$

where $\gamma\geqslant 0$, $\beta\geqslant 0$ and ν is a σ-finite positive measure on $(0,\infty)$. This integral representation is valid for the analytic continuation of f,

$$f(z)=\gamma+\beta z+\int_{0}^{\infty}z(z+t)^{-1}\,d\nu(t) \qquad \text{for} \quad \operatorname{Im} z\neq 0.$$

Consider the open sector Π with vertex 0 and the positive real axis and the half line $\arg z=p\pi$ as it boundaries. If $\operatorname{Im} z>0$, then $\gamma+\beta z^{p}$ and $z^{p}(z^{p}+t)^{-1}$ for $t>0$ belong to Π, and hence so does $f(z^{p})$ by the integral representation. Now the function $w\mapsto w^{1/p}$ is well defined and analytic on Π and maps it into the open upper half plane. Therefore the function $f(z^{p})^{1/p}$ is analytic on the open upper half plane and maps it into itself. Since the same is true on the open lower half plane, $f(\lambda^{p})^{1/p}$ is operator-monotone on $(0,\infty)$ by Lemma 3. ■

Remark. In contrast to Corollary 4.3 as well as Corollary 3.2, if Φ is a normalized positive linear map, the map $A\mapsto\Phi(A^{1/2})^{2}$ is not necessarily concave.

4. *Tensor product maps*

An important way of putting matrices together is to construct their *tensor product* (sometimes called the Kronecker product). If $A=(a_{ij})$ and $B=(b_{ij})$ are square matrices of order n and m, respectively, then their tensor product is

$$A\otimes B\equiv\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

The matrix $A\otimes B$ is an nm-square matrix.

The following formulas for tensor products are well known (see [13, p. 8]):

$$(A\otimes B)\cdot(C\otimes D)=(AC)\otimes(BD)$$

and

$$(A\otimes B)^{*}=A^{*}\otimes B^{*}.$$

Since $I_n \otimes I_m = I_{nm}$,

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

whenever both A and B are invertible. These imply that if both A and B are positive definite, so is their tensor product. More generally $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$ imply $A_1 \otimes B_1 \geqslant A_2 \otimes B_2$.

Recalling the definition of geometric mean in Sec. 1, we obtain that if A_i and B_i are positive definite ($i = 1, 2$), then

$$(A_1 \otimes B_1) \# (A_2 \otimes B_2) = (A_1 \# A_2) \otimes (B_1 \# B_2).$$

These simple facts will be used without any reference.

We shall use, for convenience, the notation $\prod_{i=1}^{k} \otimes A_i$ for $A_1 \otimes \cdots \otimes A_k$.

In this section we present concavity and convexity theorems for certain tensor product maps. Let us start with a generalization of Corollary 1.2.

THEOREM 5. *If Φ_i is a concave map from $\mathbf{H}_{n_i}^{+}$ to $\mathbf{H}_{m_i}^{+}$ ($i = 1, \ldots, k$), then the map*

$$(A_1, \ldots, A_k) \mapsto \prod_{i=1}^{k} \otimes \Phi_i(A_i)^{-1}$$

is convex.

Proof. Since the map under consideration is continuous, it suffices to show its midpoint convexity. Take A_i and B_i in $\mathbf{H}_{n_i}^{+}$ ($i = 1, \ldots, k$). Then

$$\begin{aligned}
&\prod_{i=1}^{k} \otimes \Phi_i\left(\tfrac{1}{2}(A_i + B_i)\right)^{-1} \\
&\qquad \leqslant \prod_{i=1}^{k} \otimes \left\{ \tfrac{1}{2}(\Phi_i(A_i) + \Phi_i(B_i)) \right\}^{-1} \quad \text{(by concavity of } \Phi_i\text{)} \\
&\qquad \leqslant \prod_{i=1}^{k} \otimes \left\{ \Phi_i(A_i) \# \Phi_i(B_i) \right\}^{-1} \qquad \text{(by Corollary 2.1)} \\
&\qquad = \prod_{i=1}^{k} \otimes \left\{ \Phi_i(A_i)^{-1} \# \Phi_i(B_i)^{-1} \right\} \\
&\qquad = \left\{ \prod_{i=1}^{k} \otimes \Phi_i(A_i)^{-1} \right\} \# \left\{ \prod_{i=1}^{k} \otimes \Phi_i(B_i)^{-1} \right\} \\
&\qquad \leqslant \frac{1}{2} \left\{ \prod_{i=1}^{k} \otimes \Phi_i(A_i)^{-1} + \prod_{i=1}^{k} \otimes \Phi_i(B_i)^{-1} \right\} \qquad \text{(by Corollary 2.1).}
\end{aligned}$$

■

COROLLARY 5.1 (Lieb [11]). *If $0<p_i\leqslant 1$ $(i=1,2,\dots,k)$, then the map*

$$(A_1,\dots,A_k)\mapsto \prod_{i=1}^{k}\otimes A_i^{-p_i}$$

is convex on $\prod_{i=1}^{k}\mathbf{H}_{n_i}^{+}$.

This follows immediately from Theorem 5 and Corollary 4.1.

Before stating one of our main results, we remark that for invertible Hermitian A and Hermitian B, the matrix $A\otimes I$ (I being the identity matrix) commutes with $A^{-1}\otimes B$, and hence with $f[A^{-1}\otimes B]$ if the latter is defined. Therefore $f[A^{-1}\otimes B]\cdot(A\otimes I)$ becomes a Hermitian matrix.

THEOREM 6. *Let f be a positive operator-monotone function on $(0,\infty)$. If Φ_i is a concave map from $\mathbf{H}_{n_i}^{+}$ to $\mathbf{H}_{m_i}^{+}$ $(i=1,2)$, then the map*

$$(A_1,A_2)\mapsto f\left[\Phi_1(A_1)^{-1}\otimes\Phi_2(A_2)\right]\cdot\left(\Phi_1(A_1)\otimes I_{m_2}\right)$$

is concave.

Proof. As in the proof of Corollary 4.3, f admits an integral representation:

$$f(\lambda)=\gamma+\beta\lambda+\int_0^{\infty}\lambda(\lambda+t)^{-1}\,d\nu(t),$$

where $\gamma,\beta\geqslant 0$ and ν is a positive measure on $(0,\infty)$. Then

$$\begin{aligned}
&f\left[\Phi_1(A_1)^{-1}\otimes\Phi_2(A_2)\right]\cdot\left(\Phi_1(A_1)\otimes I_{m_2}\right)\\
&\qquad=\gamma\Phi_1(A_1)\otimes I_{m_2}+\beta I_{m_1}\otimes\Phi_2(A_2)\\
&\qquad\quad+\int_0^{\infty}\left\{I_{m_1}\otimes\Phi_2(A_2)\right\}\left\{\Phi_1(A_1)^{-1}\otimes\Phi_2(A_2)+tI_{m_1}\otimes I_{m_2}\right\}^{-1}d\nu(t).
\end{aligned}$$

Since concavity of Φ_i $(i=1,2)$ together with $\gamma,\beta\geqslant 0$ implies the concavity of the map

$$(A_1,A_2)\mapsto\gamma\Phi_1(A_1)\otimes I_{m_2}+\beta I_{m_2}\otimes\Phi_2(A_2),$$

to prove the assertion of the theorem it remains only to show, for each $t>0$, the concavity of the map

$$(A_1,A_2)\mapsto\{I_{m_1}\otimes\Phi_2(A_2)\}\{\Phi_1(A_1)^{-1}\otimes\Phi_2(A_2)+tI_{m_1}\otimes I_{m_2}\}^{-1}.$$

But the right hand side is equal to

$$\{\Phi_1(A_1)^{-1}\otimes I_{m_2}+tI_{m_1}\otimes\Phi_2(A_2)^{-1}\}^{-1}=\tfrac{1}{2}\{\Phi_1(A_1)\otimes I_{m_2}\}\,!\,\{t^{-1}I_{m_1}\otimes\Phi_2(A_2)\}.$$

Now since both the maps $A_1\mapsto\Phi_1(A_1)\otimes I_{m_2}$ and $A_2\mapsto t^{-1}I_{m_1}\otimes\Phi_2(A_2)$ are concave as Φ_i $(i=1,2)$, the expected concavity follows from Corollary 2.2. ∎

COROLLARY 6.1. *Let f be a positive operator-monotone function on $(0,\infty)$. If Φ_i is a concave map from $\mathbf{H}_{n_i}^+$ to $\mathbf{H}_{m_i}^+$ $(i=1,2)$, then the map*

$$(A_1,A_2)\mapsto f[\Phi_1(A_1)\otimes\Phi_2(A_2)^{-1}]\cdot(\Phi_1(A_1)^{-1}\otimes I_{m_2})$$

is convex.

Proof. The function $g(\lambda)\equiv f(\lambda^{-1})^{-1}$ is operator-monotone, as pointed out after Lemma 3. Then since

$$\{f[\Phi_1(A_1)\otimes\Phi_2(A_2)^{-1}]\cdot(\Phi_1(A_1)^{-1}\otimes I_{m_2})\}^{-1}$$
$$=g[\Phi_1(A_1)^{-1}\otimes\Phi_2(A_2)]\cdot(\Phi_1(A_1)\otimes I_{m_2})$$

and the map

$$(A_1,A_2)\mapsto g[\Phi_1(A_1)^{-1}\otimes\Phi_2(A_2)]\cdot(\Phi_1(A_1)\otimes I_{m_2})$$

is concave by Theorem 6, the assertion follows from Corollary 1.2. ∎

COROLLARY 6.2 (Lieb [11]). *If $0\leqslant p_i\leqslant 1$ $(i=1,\ldots,k)$ and $\Sigma_{i=1}^k p_i\leqslant 1$, then the map*

$$(A_1,\ldots,A_k)\mapsto\prod_{i=1}^k\otimes A_i^{p_i}$$

is concave on $\Pi_{i=1}^k\mathbf{H}_{n_i}^+$.

Proof. The assertion is just Corollary 4.1 if $k=1$. Suppose that the assertion is generally true for the case $k-1$. If $p_k=1$, then $p_i=0$ $(i=1,\dots,k-1)$ and the map becomes

$$(A_1,\dots,A_k)\mapsto I_{m_1}\otimes\cdots\otimes I_{m_{k-1}}\otimes A_k,$$

which is trivially concave. If $p_k=0$, the map becomes

$$(A_1,\dots,A_k)\mapsto A_1^{p_1}\otimes\cdots\otimes A_k^{p_{k-1}}\otimes I_{m_k},$$

which is concave by the induction assumption. Now suppose $0<p_k<1$. Then the map Φ

$$(A_1,\dots,A_{k-1})\mapsto \prod_{i=1}^{k-1}\otimes A_i^{p_i/(1-p_k)}$$

is concave by the induction assumption. Now just as in the proof of Theorem 6 with $f(\lambda)=\lambda^{p_k}$, the map

$$\begin{aligned}(A_1,\dots,A_k)&\mapsto f\left[\Phi(A_1,\dots,A_{k-1})^{-1}\otimes A_k\right]\cdot\left(\Phi(A_1,\dots,A_{k-1})\otimes I_{m_k}\right)\\&=\prod_{i=1}^{k}\otimes A_i^{p_i}\end{aligned}$$

is concave. ■

Corollary 6.3. *If* $1\leqslant q\leqslant 2$, $0\leqslant p_i\leqslant 1$ $(i=1,2,\dots,k)$ *and* $\sum_{i=1}^{k}p_i\leqslant q-1$, *then the map*

$$(A_0,A_1,\dots,A_k)\mapsto A_0^{q}\otimes\left(\prod_{i=1}^{k}\otimes A_i^{-p_i}\right)$$

is convex on $\prod_{i=0}^{k}H_{n_i}^{+}$.

Proof. By Corollary 6.2, the map

$$\Phi(A_0,A_1,\dots,A_k)=A_0^{2-q}\otimes\left\{\prod_{i=1}^{k}\otimes A_i^{p_i}\right\}$$

is concave, while the map

$$\Psi(A_0, A_1, \ldots, A_k) = A_0 \otimes \left\{ \prod_{i=1}^{k} \otimes I_{m_i} \right\}$$

is affine. Then the assertion follows just as for Corollary 1.2. ■

REMARKS. (1) Our approach in the proof of Theorem 6 is akin to that of Epstein [9].

(2) Corollary 6.2 can be proved directly by using geometric means as in Uhlman [17]. In fact, with $k=2$, the set Δ of (p_1, p_2) for which the map $(A_1, A_2) \mapsto A_1^{p_1} \otimes A_2^{p_2}$ is concave is convex by Corollary 2.2. It contains $(1,0)$ and $(0,1)$, and hence all $(p, 1-p)$ for $0 \leqslant p \leqslant 1$.

(3) Corollary 6.3 seems not to have been noticed before.

(4) Let $-\infty < s, t < \infty$. The map $A \mapsto A^s \otimes A^t$ is concave if and only if $0 \leqslant s, t$ and $s + t \leqslant 1$. It is convex if and only if (a) $-1 \leqslant s, t \leqslant 0$, or (b) $-1 \leqslant s \leqslant 0$ and $1 - s \leqslant t \leqslant 2$, or (c) $-1 \leqslant t \leqslant 0$ and $1 - t \leqslant s \leqslant 2$.

The following theorem gives a generalization of Corollary 6.3 and presents a convexity theorem of new type.

THEOREM 7. *Let f be an operator-monotone function on $(0, \infty)$. If Φ_1 is an affine map from $\mathbf{H}_{n_1}^+$ to $\mathbf{H}_{m_1}^+$ and if Φ_2 is a concave map from $\mathbf{H}_{n_2}^+$ to $\mathbf{H}_{m_2}^+$, then the map*

$$(A_1, A_2) \mapsto f\left[\Phi_1(A_1) \otimes \Phi_2(A_2)^{-1}\right] \cdot \left(\Phi_1(A_1) \otimes I_{m_2}\right)$$

is convex.

Proof. By Lemma 3, f admits an integral representation

$$f(\lambda) = \alpha + \beta\lambda - \gamma\lambda^{-1} + \int_{-\infty}^{0-} (1 + \lambda t)(t - \lambda)^{-1} d\mu(t),$$

where α is real, β and γ are non-negative and μ is a positive measure on $(-\infty, 0)$. Then

$$\begin{aligned} f&\left[\Phi_1(A_1) \otimes \Phi_2(A_2)^{-1}\right] \cdot \left(\Phi_1(A_1) \otimes I_{m_2}\right) \\ &= \alpha \Phi_1(A_1) \otimes I_{m_2} + \beta \Phi_1(A_1)^2 \otimes \Phi_2(A_2)^{-1} - \gamma I_{m_1} \otimes \Phi_2(A_2) \\ &+ \int_{-\infty}^{0-} \left\{ \Phi_1(A_1) \otimes I_{m_2} + t \Phi_1(A_1)^2 \otimes \Phi_2(A_2)^{-1} \right\} \\ &\times \left\{ t I_{m_1} \otimes I_{m_2} - \Phi_1(A_1) \otimes \Phi_2(A_2)^{-1} \right\}^{-1} d\mu(t). \end{aligned}$$

Since Φ_1 is affine and Φ_2 is concave by assumption, the map

$$(A_1,A_2)\mapsto\alpha\Phi_1(A_1)\otimes I_{m_2}-\gamma I_{m_1}\otimes\Phi_2(A_2)$$

is convex. Since for each $t<0$

$$\begin{aligned}\left\{\Phi_1(A_1)\otimes I_{m_2}+t\Phi_1(A_1)^2\otimes\Phi_2(A_2)^{-1}\right\}&\left\{tI_{m_1}\otimes I_{m_2}-\Phi_1(A_1)\otimes\Phi_2(A_2)^{-1}\right\}^{-1}\\ &=-t\Phi_1(A_1)\otimes I_{m_2}-(1+t^2)\left\{I_{m_1}\otimes\Phi_2(A_2)^{-1}+|t|\Phi_1(A_1)^{-1}\otimes I_{m_2}\right\}^{-1}\\ &=-t\Phi_1(A_1)\otimes I_{m_2}-2^{-1}(1+t^2)\left\{\left(I_{m_1}\otimes\Phi_2(A_2)\right)!\left(|t|^{-1}\Phi_1(A_1)\otimes I_{m_2}\right)\right\},\end{aligned}$$

Corollary 2.2 implies the convexity of the map

$$\begin{aligned}(A_1,A_2)\mapsto\int_{-\infty}^{0-}&\left\{\Phi_1(A_1)\otimes I_{m_2}+t\Phi_1(A_1)^2\otimes\Phi_2(A_2)^{-1}\right\}\\ &\times\left\{tI_{m_1}\otimes I_{m_2}-\Phi_1(A_1)\otimes\Phi_2(A_2)^{-1}\right\}^{-1}d\mu(t).\end{aligned}$$

Finally the map

$$(A_1,A_2)\mapsto\Phi_1(A_1)^2\otimes\Phi_2(A_2)^{-1}$$

is convex by Corollary 1.2, because

$$\Phi_1(A_1)^2\otimes\Phi_2(A_2)^{-1}=\left(\Phi_1(A_1)\otimes I_{m_2}\right)\left(I_{m_1}\otimes\Phi_2(A_2)\right)^{-1}\left(\Phi_1(A_1)\otimes I_{m_2}\right).\quad\blacksquare$$

COROLLARY 7.1. *The map*

$$(A_1,A_2)\mapsto(A_1\log[A_1])\otimes I_{n_2}-A_1\otimes\log[A_2]$$

is convex on $\mathbf{H}_{n_1}^+\times\mathbf{H}_{n_2}^+$.

Proof. Since $\log\lambda$ is operator-monotone, Theorem 7 yields the convexity of the map

$$(A_1,A_2)\mapsto\log\left[A_1\otimes A_2^{-1}\right]\cdot\left(A_1\otimes I_{n_2}\right).$$

Now the assertion follows from the identity

$$\log[A_1 \otimes A_2^{-1}] = \log\left[(A_1 \otimes I_{n_2})(I_{n_1} \otimes A_2)^{-1}\right]$$
$$= \log[A_1] \otimes I_{n_2} - I_{n_1} \otimes \log[A_2]. \qquad \blacksquare$$

REMARK. (1) When $f(\lambda) = \lambda^p$ with $0 < p < 1$, Theorem 7 reduces to Corollary 6.3 with $k = 2$ and $q = 1 + p$.

(2) Corollary 7.1 can be obtained from Corollary 6.2 by differentiating $A_1^{1-p} \otimes A_2^p$ at $p = 0$.

APPLICATION

5. *Hadamard products*

In this and subsequent sections all matrices are n-square with fixed n, and I stands for the identity matrix. For $A = (a_{ij})$ and $B = (b_{ij})$, their *Hadamard product* is the n-square matrix of entrywise products

$$A * B \equiv (a_{ij} b_{ij}).$$

The Hadamard product differs from the usual product in many ways. The most important is commutativity of Hadamard multiplication:

$$A * B = B * A.$$

The diagonal matrix formed from a matrix A can be obtained by Hadamard multiplication with the identity matrix $A * I$. The following convenient rule will be used repeatedly in subsequent sections:

$$(A * B) * I = (A * I)(B * I).$$

A nice survey on Hadamard products can be found in Styan [15].

We shall use, for convenience, the notation

$$\prod_{i=1}^{k} * A_i = A_1 * \cdots * A_k$$

and

$$\prod_{1}^{k} *A = A * \cdots * A \quad (k \text{ times}).$$

It has been known for some time that the Hadamard product is a principal n-square submatrix of the tensor product $A \otimes B$. This can be found in Marcus and Khan [12], and even in Tôyama's book [16, p. 199], published in 1952. We formulate this fact as follows:

LEMMA 4. *For each positive integer k there is a normalized positive linear map Φ_k from $\mathbf{H}_{n^k}$ to $\mathbf{H}_n$ such that*

$$\Phi_k\left(\prod_{i=1}^{k} \otimes A_i\right) = \prod_{i=1}^{k} *A_i \qquad \textit{for all } A_i.$$

We shall present simple consequences of this lemma. Some of them have been known but not presented in these forms.

THEOREM 8. *If $-B_i \leqslant A_i \leqslant B_i$ $(i = 1, \dots, k)$, then*

$$-\prod_{i=1}^{k} *B_i \leqslant \prod_{i=1}^{k} *A_i \leqslant \prod_{i=1}^{k} *B_i.$$

Proof. This follows from Lemma 4, because the inequalities

$$-\prod_{i=1}^{k} \otimes B_i \leqslant \prod_{i=1}^{k} \otimes A_i \leqslant \prod_{i=1}^{k} \otimes B_i$$

can be proved by induction on k. ■

COROLLARY 8.1 (Schur). *If $0 \leqslant A_i \leqslant B_i$ $(i = 1, \dots, k)$, then*

$$0 \leqslant \prod_{i=1}^{k} *A_i \leqslant \prod_{i=1}^{k} *B_i.$$

If A_i is positive definite $(i = 1, \dots, k)$, so is $\prod_{i=1}^{k} *A_i$.

Recall that for Hermitian A the positive semidefinite square root of A^2 is denoted by $|A|$. Then $-|A| \leqslant A \leqslant |A|$.

COROLLARY 8.2. *If A_i are Hermitian $(i=1,\dots,k)$, then*

$$-\prod_{i=1}^{k} *|A_i| \leqslant \prod_{i=1}^{k} *A_i \leqslant \prod_{i=1}^{k} *|A_i|.$$

REMARK. (1) Corollaries 8.1 and 8.2 are implicit in [15].

(2) Corollary 8.2 cannot be improved to the form (see [15], p. 232)

$$\left|\prod_{i=1}^{k} *A_i\right| \leqslant \prod_{i=1}^{k} *|A_i|.$$

THEOREM 9. *If A_i is positive semidefinite $(i=1,\dots,k)$, then*

$$\prod_{i=1}^{k} *A_i \leqslant \frac{k^k}{k!} \prod_{1}^{k} *\left(k^{-1}\sum_{i=1}^{k} A_i\right).$$

Here the constant $k^k/k!$ is sharp if n is large.

Proof. Since $\prod_{i=1}^{k} *A_{\sigma_i}$ is invariant for any permutation σ of the indices $\{1,2,\dots,k\}$, by Corollary 8.1 the inequality in the theorem follows from the inequality

$$\prod_{1}^{k} *\left(\sum_{i=1}^{k} A_i\right) \geqslant \sum_{\sigma}\left(\prod_{i=1}^{k} *A_{\sigma_i}\right),$$

where σ runs over the set of all permutations.

Let $m=2^k$, and label elements of the set $\{1,2,\dots,m\}$ by points of the set $\Omega \equiv \{-1,1\}^k$. For each $i=1,\dots,k$ consider an m-square matrix A_i whose (ω,ω')-entry, with ω,ω' in Ω, is given by $\omega_i\omega_i'$, where ω_i is the ith coordinate of ω. Obviously A_i is positive semidefinite. Then for each ω,ω' in Ω the (ω,ω')-entry of $\prod_1^k *(\sum_{i=1}^k A_i)$ is equal to

$$\sum_{\tau}\left(\prod_{i=1}^{k} \omega_{\tau_i}\right)\left(\prod_{i=1}^{k} \omega'_{\tau_i}\right),$$

where τ runs over the set of all maps from $\{1,2,\ldots,k\}$ into itself. For any such τ there is $j\leqslant k$ and an injective map σ from $\{1,2,\ldots,j\}$ into $\{1,2,\ldots,k\}$ such that

$$\prod_{i=1}^{k}\omega_{\tau_i}=\prod_{i=1}^{j}\omega_{\sigma_i}\qquad\text{for all}\quad\omega\text{ in }\Omega.$$

This consideration shows that

$$\prod_{1}^{k}*\left(\sum_{i=1}^{k}A_i\right)=k!\prod_{i=1}^{k}*A_i+\sum_{j=1}^{k-1}\sum_{\sigma}\alpha_\sigma\prod_{i=1}^{j}*A_{\sigma_i},$$

where $\alpha_\sigma\geqslant 0$ and σ runs over the set of all injective maps from $\{1,2,\ldots,j\}$ into $\{1,2,\ldots,k\}$. For each such σ the (ω,ω')-entry of

$$\left(\prod_{i=1}^{j}*A_{\sigma_i}\right)\left(\prod_{i=1}^{k}*A_i\right)$$

is equal to

$$\left(\prod_{i=1}^{j}\omega_{\sigma_i}\right)\left\{\sum_{\omega''}\left(\prod_{i=1}^{j}\omega''_{\sigma_i}\right)\left(\prod_{i=1}^{k}\omega''_i\right)\right\}\left(\prod_{i=1}^{k}\omega'_i\right).$$

Since $j<k$ and σ is injective, there is $0<j'<k$ and an injective map σ' from $\{1,\ldots,j'\}$ into $\{1,2,\ldots,k\}$ such that

$$\left(\prod_{i=1}^{j}\omega''_{\sigma_i}\right)\left(\prod_{i=1}^{k}\omega''_i\right)=\prod_{i=1}^{j'}\omega''_{\sigma'_i}\qquad\text{for all}\quad\omega''\text{ in }\Omega.$$

Then it is readily shown that

$$\sum_{\omega''}\left(\prod_{i=1}^{j}\omega''_{\sigma_i}\right)\left(\prod_{i=1}^{k}\omega''_i\right)=0$$

and consequently

$$\left(\prod_{i=1}^{j}*A_{\sigma_i}\right)\left(\prod_{i=1}^{k}*A_i\right)=0.$$

Finally, to see the sharpness of the constant $k^k/k!$, suppose that, for the above-constructed A_i,

$$\prod_1^k * \left(k^{-1}\sum_{i=1}^k A_i\right) \geqslant \gamma \prod_{i=1}^k * A_i.$$

Then it follows that

$$k^{-k}k! \prod_{i=1}^k * A_i + k^{-k}\sum_{j=1}^{k-1}\sum_\sigma \alpha_\sigma \prod_{i=1}^j * A_{\sigma_i} \geqslant \gamma \prod_{i=1}^k * A_i.$$

Multiply by $\prod_{i=1}^k * A_i$ on the left and right to get

$$k^{-k}k!\left(\prod_{i=1}^k * A_i\right)^3 \geqslant \gamma\left(\prod_{i=1}^k * A_i\right)^3.$$

Now $k^{-k}k! \geqslant \gamma$ follows from the nonvanishing of $(\prod_{i=1}^k * A_i)^3 \geqslant 0$. ∎

6. *Inequalities of the Hölder type*

We shall apply concavity and convexity theorems developed in the first part of this paper to Hadamard products of positive definite matrices and obtain inequalities of the Hölder type.

THEOREM 10. *If A_i is positive definite $(i=1,2,\ldots,k)$, then the following estimates from above hold for the Hadamard product $\prod_{i=1}^k * A_i$:*

(i) $\displaystyle \prod_{i=1}^k * A_i \leqslant \left(\prod_{i=1}^k * A_i^p\right)^{1/p}$ *if* $1 \leqslant p < \infty$.

(ii) $\displaystyle \prod_{i=1}^k * A_i \leqslant \prod_{i=1}^k * \left(k^{-1}\sum_{j=1}^k A_j^{p_i}\right)^{1/p_i}$ *if* $p_i \geqslant 1$ *and* $\displaystyle\sum_{i=1}^k \frac{1}{p_i} = 1$.

In particular,

$$\prod_{i=1}^k * A_i \leqslant \prod_1^k * \left(k^{-1}\sum_{i=1}^k A_i^k\right)^{1/k}.$$

Proof. (i) follows immediately from Lemma 4 and Corollary 4.2. (ii) results similarly from Lemma 4 and Corollary 6.2. In fact, the map

$$(X_1,\dots,X_k)\mapsto \prod_{i=1}^{k} *X_i^{1/p_i}$$

becomes concave. Define A_i for all $i \geqslant k$ by the relation $A_i = A_j$ whenever $i \equiv j \pmod{k}$. Then the concavity combined with commutativity of Hadamard multiplication yields

$$\begin{aligned}\prod_{i=1}^{k} *A_i &= k^{-1}\sum_{j=0}^{k-1}\prod_{i=1}^{k} *A_{i+j}\\ &= k^{-1}\sum_{j=0}^{k-1}\prod_{i=1}^{k} *(A_{i+j}^{p_i})^{1/p_i}\\ &\leqslant \prod_{i=1}^{k} *\left(k^{-1}\sum_{j=0}^{k-1} A_{i+j}^{p_i}\right)^{1/p_i}\\ &= \prod_{i=1}^{k} *\left(k^{-1}\sum_{j=1}^{k} A_j^{p_i}\right)^{1/p_i}. \qquad \blacksquare\end{aligned}$$

THEOREM 11. *If A_i is positive definite $(i=1,2,\dots,k)$, then the following estimates from below hold for the Hadamard product $\prod_{i=1}^{k} *A_i$:*

(i) $\displaystyle \prod_{i=1}^{k} *A_i \geqslant \left(\prod_{i=1}^{k} *A_i^{p}\right)^{1/p}$ *if* $\frac{1}{2} \leqslant p \leqslant 1$.

(ii) $\displaystyle \prod_{i=1}^{k} *A_i \geqslant \left(\prod_{i=1}^{k} *A_i^{-p}\right)^{-1/p}$ *if* $1 \leqslant p < \infty$.

(iii) $\displaystyle \prod_{i=1}^{k} *A_i \geqslant \prod_{1}^{k} *\left(k^{-1}\sum_{j=1}^{k} A_j^{-p}\right)^{-1/p}$ *if* $1 \leqslant p < \infty$.

(iv) $\displaystyle \prod_{i=1}^{k} *A_i \geqslant \left(k^{-1}\sum_{i=1}^{k} A_i^{q}\right)^{1/q} * \left\{\prod_{i=1}^{k-1} *\left(k^{-1}\sum_{j=1}^{k} A_j^{-p_i}\right)^{-1/p_i}\right\}$ *if* $q,p_i >$ 0 *and* $\displaystyle \frac{1}{q} - 1 = \sum_{i=1}^{k-1}\frac{1}{p_i} \leqslant 1.$

In particular,

$$\prod_{i=1}^{k} * A_i \geqslant \left(k^{-1}\sum_{i=1}^{k} A_i^{1/2}\right)^2 * \left\{\prod_{1}^{k-1} * \left(k^{-1}\sum_{i=1}^{k} A_i^{1-k}\right)^{1/(1-k)}\right\}.$$

Proof. (i) and (ii) follow immediately from Lemma 4 and Corollary 4.2. As in the proof of Theorem 9, (iii) and (iv) result from the convexity of the maps

$$(X_1,\dots,X_k)\mapsto \prod_{i=1}^{k} * X_i^{-1/p} \qquad \text{if} \quad 1\leqslant p<\infty$$

and

$$(X_1,\dots,X_k)\mapsto X_1^{1/q} * \left(\prod_{i=2}^{k} * X_i^{-1/p_i}\right) \qquad \text{if} \quad \frac{1}{q}-1=\sum_{i=2}^{k}\frac{1}{p_i}\leqslant 1,$$

assured by Corollary 5.1 and Corollary 6.3, respectively. ∎

THEOREM 12. *If positive definite A_i $(i=1,2,\dots,k)$ commute, then*

$$\prod_{i=1}^{k} * A_i \geqslant \prod_{1}^{k} * \left(\prod_{i=1}^{k} A_i\right)^{1/k}.$$

Proof. Let $p_i = i/(i+1)$ $(i=1,2,\dots,k)$, and $Y=\prod_{i=1}^{k}\otimes A_{i+1}$. Consider the normalized positive linear map Ψ from $\mathbf{H}_{n^k}$ to $\mathbf{H}_n$.

$$\Psi(X)\equiv\Phi_k(Y)^{-1/2}\Phi_k(Y^{1/2}XY^{1/2})\Phi_k(Y)^{-1/2}.$$

Then by Theorem 4, $\Psi(X^{p_1})\leqslant\Psi(X)^{p_1}$ for all $X\geqslant 0$. Since the A_i commute, with $X\equiv\prod_{i=1}^{k}\otimes(A_iA_{i+1}^{-1})$ $(A_{k+1}=A_1)$ the above inequality yields

$$\left(\prod_{i=2}^{k+1} * A_i\right)^{-1/2}\left\{\prod_{i=1}^{k} * (A_i^{p_1}A_{i+1}^{1-p_1})\right\}\left(\prod_{i=2}^{k+1} * A_i\right)^{-1/2}$$

$$\leqslant\left\{\left(\prod_{i=1}^{k} * A_i\right)^{-1/2}\left(\prod_{i=1}^{k} * A_i\right)\left(\prod_{i=1}^{k} * A_i\right)^{-1/2}\right\}^{p_1}=I;$$

hence

$$\prod_{i=1}^{k} *\left(A_i^{p_1} A_{i+1}^{1-p_1}\right) \leqslant \prod_{i=1}^{k} *A_i.$$

Analogous arguments will show, with $A_{k+2}=A_2$,

$$\prod_{i=1}^{k} *\left(\left(A_i^{p_1} A_{i+1}^{1-p_1}\right)^{p_2} A_{i+2}^{1-p_2}\right) \leqslant \prod_{i=1}^{k} *A_i.$$

Since

$$\prod_{i=1}^{k-1} p_i = (1-p_j)\prod_{i=j+1}^{k-1} p_i = 1-p_{k-1} = \frac{1}{k} \qquad (j=1,\ldots,k-1),$$

the above consideration will finally lead to

$$\prod_{1}^{k} *\left(\prod_{i=1}^{k} A_i^{1/k}\right) \leqslant \prod_{i=1}^{k} *A_i. \qquad \blacksquare$$

For the case $k=2$, use of the geometric mean gives better estimates from below for the Hadamard product of positive definite matrices A,B. Recall that $A \# B$ denotes the geometric mean of A and B.

Theorem 13. *If A and B are positive definite, then*

$$A*B \geqslant (A\#B)*(A\#B).$$

Proof. By commutativity of Hadamard multiplication, Lemma 4 and Theorem 3 yield

$$\begin{aligned} A*B &= \Phi_2(A\otimes B)\ \#\ \Phi_2(B\otimes A) \\ &\geqslant \Phi_2((A\otimes B)\ \#\ (B\otimes A)) \\ &= (A\#B)*(B\#A) = (A\#B)*(A\#B). \end{aligned} \qquad \blacksquare$$

COROLLARY 13.1. *If A and B are positive definite, then for any $0<\lambda<1$*

$$A*B\geqslant\{\lambda A^{-1}+(1-\lambda)B^{-1}\}^{-1}*\{(1-\lambda)A+\lambda B\}.$$

In particular,

$$A*B\geqslant\left\{\tfrac{1}{2}(A^{-1}+B^{-1})\right\}^{-1}*\left\{\tfrac{1}{2}(A+B)\right\}$$

$$=(A^{-1}+B^{-1})^{-1}*(A+B).$$

Proof. Since it is easily seen from the definition of the geometric mean that $(XY^{-1}X)\#Y=X$ for any positive definite X and Y,

$$\{\lambda A^{-1}+(1-\lambda)B^{-1}\}^{-1}*\{(1-\lambda)A+\lambda B\}$$

$$=\lambda^{-1}\left\{A-(1-\lambda)A\{(1-\lambda)A+\lambda B\}^{-1}A\right\}*\{(1-\lambda)A+\lambda B\}$$

$$=\lambda^{-1}(1-\lambda)A*A+A*B-\lambda^{-1}(1-\lambda)A\{(1-\lambda)A+\lambda B\}^{-1}A*\{(1-\lambda)A+\lambda B\}$$

$$\leqslant\lambda^{-1}(1-\lambda)A*A+A*B-\lambda^{-1}(1-\lambda)A*A=A*B,$$

where the last inequality is assured by Theorem 13. ∎

Let us show that the inequality in Corollary 13.1,

$$A*B\geqslant\left\{\tfrac{1}{2}(A^{-1}+B^{-1})\right\}^{-1}*\left\{\tfrac{1}{2}(A+B)\right\},$$

gives an improvement over the estimates from below for $A*B$ that are derived from Theorem 11:

$$A*B\geqslant\left\{\tfrac{1}{2}(A^{-p}+B^{-p})\right\}^{-1/p}*\left\{\tfrac{1}{2}(A^{-p}+B^{-p})\right\}^{-1/p}\qquad\text{if}\quad 1\leqslant p<\infty$$

and

$$A*B\geqslant\left\{\tfrac{1}{2}(A^{-p}+B^{-p})\right\}^{-1/p}*\left\{\tfrac{1}{2}(A^{q}+B^{q})\right\}^{1/q}$$

$$\text{if}\quad p,q>0\ \text{and}\ \frac{1}{q}-1=\frac{1}{p}\leqslant 1.$$

In fact, consider the normalized positive linear map Φ from $\mathbf{H}_{n^2}$ to $\mathbf{H}_n$ defined by

$$\Phi:\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \mapsto \tfrac{1}{2}(A+B),$$

where A, B and C are n-square. Then Lemma 2 and Corollary 4.2 yield the inequalities

$$\tfrac{1}{2}(A+B) \geqslant \left\{\tfrac{1}{2}(A^{-1}+B^{-1})\right\}^{-1} \geqslant \left\{\tfrac{1}{2}(A^{-p}+B^{-p})\right\}^{-1/p} \qquad \text{if} \quad 1 \leqslant p < \infty$$

and

$$\tfrac{1}{2}(A+B) \geqslant \left\{\tfrac{1}{2}(A^{q}+B^{q})\right\}^{1/q} \qquad \text{if} \quad \tfrac{1}{2} \leqslant q \leqslant 1.$$

Therefore by Corollary 8.1

$$\left\{\tfrac{1}{2}(A^{-1}+B^{-1})\right\}^{-1} * \left\{\tfrac{1}{2}(A+B)\right\} \geqslant \left\{\tfrac{1}{2}(A^{-p}+B^{-p})\right\}^{-1/p}$$

$$* \left\{\tfrac{1}{2}(A^{-p}+B^{-p})\right\}^{-1/p} \qquad \text{if} \quad 1 \leqslant p < \infty$$

and

$$\left\{\tfrac{1}{2}(A^{-1}+B^{-1})\right\}^{-1} * \left\{\tfrac{1}{2}(A+B)\right\} \geqslant \left\{\tfrac{1}{2}(A^{-p}+B^{-p})\right\}^{-1/p}$$

$$* \left\{\tfrac{1}{2}(A^{q}+B^{q})\right\}^{1/q} \qquad \text{if} \quad p,q>0 \text{ and } \frac{1}{q}-1=\frac{1}{p} \leqslant 1.$$

It is not without interest to formulate the consequences of convexity and concavity theorems in Sec. 4, which were already used in the proof of Theorem 10 and 11, in the form of inequalities of the Hölder type.

THEOREM 14. *If A_i and B_i are positive definite $(i=1,2,\ldots,k)$, the following inequalities are valid:*

(i) $\sum_{i=1}^{k}(A_i * B_i) \leqslant \left\{\sum_{i=1}^{k} A_i^p\right\}^{1/p} * \left\{\sum_{i=1}^{k} B_i^q\right\}^{1/q}$ *if $p,q \geqslant 1$ and $1/p+1/q$ $=1$.*

(ii) $\sum_{i=1}^{k}(A_i * B_i) \geqslant \left\{\sum_{i=1}^{k} A_i^{-p}\right\}^{-1/p} * \left\{\sum_{i=1}^{k} B_i^q\right\}^{1/q}$ *if $p \geqslant 1 \geqslant q \geqslant \tfrac{1}{2}$ and $1/q-1/p=1$.*

7. *Estimates by diagonal matrices*

In this section we present various estimates from above or below for Hadamard products by diagonal matrices. Recall that $A*I$ is just the diagonal matrix formed from A, and

$$\left(\prod_{i=1}^{k} *A_i\right)*I=\prod_{i=1}^{k}(A_i*I).$$

For positive definite A let us denote by ρ_A and δ_A its maximum and minimum eigenvalues, respectively. Then since

$$\rho_A I \geqslant A \geqslant \delta_A I,$$

the following estimates, which are implicit in Davis [7] and Styan [15], follow immediately from Corollary 8.1.

THEOREM 15. *If A_i is positive definite $(i=1,2,\dots,k)$, then*

$$\delta_{A_1}\prod_{i=2}^{k}(A_i*I)\leqslant \prod_{i=1}^{k} *A_i \leqslant \rho_{A_1}\prod_{i=2}^{k}(A_i*I).$$

Since

$$\lim_{p\to\infty}(A_1^p*I)^{1/p}\leqslant \rho_{A_1}I,$$

the following theorem can be considered a generalization of a part of Theorem 15.

THEOREM 16. *If A_i is positive definite $(i=1,2,\dots,k,\ k\geqslant 2)$ and if $p,q\geqslant 1$ and $1/p+1/q=1$, then*

$$\prod_{i=1}^{k} *A_i\leqslant (A_1^p*I)^{1/p}(A_2^q*I)^{1/q}\prod_{i=3}^{k}(A_i*I).$$

In particular,

$$\prod_{i=1}^{k} *A_i\leqslant (A_1^2*I)^{1/2}(A_2^2*I)^{1/2}\prod_{i=3}^{k}(A_i*I).$$

Proof. We may confine to the case $k=2$; for positive definite A, B

$$A * B \leqslant (A^p * I)^{1/p}(B^q * I)^{1/q}.$$

To prove this, let

$$C \equiv (A^p * I)^{1/p} \equiv \operatorname{diag}(\lambda_i) \quad \text{and} \quad D \equiv (B^q * I)^{1/q} \equiv \operatorname{diag}(\mu_i).$$

We may assume that $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ for $i \neq j$. Then by Theorem 14, for any $\varepsilon > 0$

$$C * D + \varepsilon A * B \leqslant (C^p + \varepsilon A^p)^{1/p} * (D^q + \varepsilon B^q)^{1/q},$$

and hence

$$A * B \leqslant \frac{d}{d\varepsilon}(C^p + \varepsilon A^p)^{1/p} * (D^q + \varepsilon B^q)^{1/q}\bigg|_{\varepsilon=0}.$$

It is easy to see (e.g. [6]) that

$$\frac{d}{d\varepsilon}(C^p + \varepsilon A^p)^{1/p}\bigg|_{\varepsilon=0} = X * A^p$$

and

$$\frac{d}{d\varepsilon}(D^q + \varepsilon B^q)^{1/q}\bigg|_{\varepsilon=0} = Y * B^q,$$

where $X \equiv (\xi_{ij})$ and $Y \equiv (\eta_{ij})$ are defined as follows:

$$\xi_{ij} = (\lambda_i - \lambda_j)(\lambda_i^p - \lambda_j^p)^{-1} \quad \text{for } i \neq j \quad \text{and} \quad \xi_{ii} = p^{-1}\lambda_i^{1-p},$$

and

$$\eta_{ij} = (\mu_i - \mu_j)(\mu_i^q - \mu_j^q)^{-1} \quad \text{for } i \neq j \quad \text{and} \quad \eta_{ii} = q^{-1}\mu_i^{1-q}.$$

Substitution of these expressions into the above inequality will yield

$$\begin{aligned} A * B &\leqslant X * A^p * D + C * Y * B^q = (X * I)(A^p * I)D + C(Y * I)(B^q * I) \\ &= p^{-1}C^{1-p}(A^p * I)D + q^{-1}CD^{1-q}(B^q * I) \\ &= (A^p * I)^{1/p}(B^q * I)^{1/q}. \end{aligned}$$

■

COROLLARY 16.1. *If A_i is positive definite ($i=1,2,\ldots,k$, $k\geqslant 2$) and if $p_i\geqslant 1$ and $\sum_{i=1}^{k}1/p_i\leqslant 1$, then*

$$\prod_{i=1}^{k} * A_i \leqslant \prod_{i=1}^{k}\left(A_i^{p_i} * I\right)^{1/p_i}.$$

In particular,

$$\prod_{i=1}^{k} * A_i \leqslant \prod_{i=1}^{k}\left(A_i^{k} * I\right)^{1/k}.$$

Proof. Let $q=p_1/(p_1-1)$. Then $p_2\geqslant q$ by assumption. Corollary 4.2 and Lemma 4 imply

$$A_2^q * I\leqslant \left(A_2^{p_2} * I\right)^{q/p_2} \quad \text{and} \quad A_i * I\leqslant \left(A_i^{p_i} * I\right)^{1/p_i} \quad (i=3,4,\ldots,k).$$

Since the function $\lambda^{1/q}$ is operator-monotone, the first inequality implies

$$\left(A_2^q * I\right)^{1/q}\leqslant \left(A_2^{p_2} * I\right)^{1/p_2}.$$

Now the assertion of the theorem follows from Theorem 16 and Corollary 8.1. ∎

REMARK. (1) Use of differentiation in a proof of a matrix inequality of this kind has already been made by Wigner and Yanase [18], who proved the concavity of the map $(A,B)\mapsto A^{1/2}\otimes B^{1/2}$.

(2) The inequality

$$A * B\leqslant \left(A^2 * I\right)^{1/2}\left(B^2 * I\right)^{1/2}$$

can be proved by using Gersgorin's theorem (see [13, p. 146]). Since

$$\lim_{p\to\infty}\left(A_1^{-p} * I\right)^{-1/p}\geqslant \delta_{A1} I,$$

the following theorem can be considered a generalization of a part of Theorem 15.

THEOREM 17. *If A_i is positive definite* ($i=1,2,\ldots,k,\ k\geqslant 2$) *and if* $p\geqslant 1\geqslant q\geqslant \frac{1}{2}$ *and* $1/q-1/p=1$, *then*

$$\prod_{i=1}^{k} *A_i \geqslant (A_1^{-p}*I)^{-1/p}(A_2^{q}*I)^{1/q}\prod_{i=3}^{k}(A_i*I).$$

In particular,

$$\prod_{i=1}^{k} *A_i \geqslant (A_1^{-1}*I)(A_2^{1/2}*I)^2\prod_{i=3}^{k}(A_i*I).$$

Proof. This theorem can be derived from Theorem 14 through the differentiation process used for Theorem 16. ■

Oppenheim (see e.g. [15, p. 227]) proved the following determinant inequality for the Hadamard product of positive definite matrices A_i:

$$\det\left(\prod_{i=1}^{k} *A_i\right) \geqslant \prod_{i=1}^{k}\det(A_i).$$

Since for any positive definite A

$$\begin{aligned}\log(\det(A)) &= \mathrm{trace}(\log[A])\\ &= \mathrm{trace}(\log[A]*I),\end{aligned}$$

the following theorem can be considered a matricial generalization of Oppenheim's inequality.

THEOREM 18. *If A_i is positive definite* ($i=1,2,\ldots,k,\ k\geqslant 2$), *then*

$$\log\left[\prod_{i=1}^{k} *A_i\right] \geqslant \left(\sum_{i=1}^{k}\log[A_i]\right)*I.$$

Proof. By Corollary 4.2 and Lemma 4,

$$\begin{aligned}\log\left[\prod_{i=1}^{k} *A_i\right] &= \log\left[\Phi_k\left(\prod_{i=1}^{k}\otimes A_i\right)\right]\\ &\geqslant \Phi_k\left(\log\left[\prod_{i=1}^{k}\otimes A_i\right]\right).\end{aligned}$$

Since

$$\log\left[\prod_{i=1}^{k} \otimes A_i\right] = \sum_{i=1}^{k} \left(I\otimes\cdots\otimes I\otimes\log[A_i]\otimes I\otimes\cdots\otimes I\right),$$

we have

$$\Phi_k\left(\log\left[\prod_{i=1}^{k}\otimes A_i\right]\right) = \sum_{i=1}^{k}\log[A_i]*I$$

$$=\left(\sum_{i=1}^{k}\log[A_i]\right)*I. \qquad \blacksquare$$

Corollary 18.1. *If B_i is Hermitian $(i=1,2,\dots,k,\ \ k\geqslant 2)$, then $(\sum_{i=1}^{k}B_i)*I\geqslant 0$ implies $\prod_{i=1}^{k}*\exp[B_i]\geqslant I$.*

Proof. Apply Theorem 18 with $A_i\equiv\exp[B_i]$ to get

$$\log\left[\prod_{i=1}^{k}*\exp[B_i]\right]\geqslant\left(\sum_{i=1}^{k}B_i\right)*I\geqslant 0,$$

which leads to the assertion. $\blacksquare$

Corollary 18.2 (Fiedler [10]). *If A is positive definite, then $A*A^{-1}\geqslant I$.*

Proof. Apply Corollary 18.1 with $B_1=\log[A]$ and $B_2=\log[A^{-1}]$. $\blacksquare$

Remark. Corollary 18.2 can be derived by using Lemma 2. In fact,

$$A*A^{-1}=\Phi_2(A\otimes A^{-1})\geqslant\Phi_2(A^{-1}\otimes A)^{-1}$$

$$=\left(A^{-1}*A\right)^{-1}=\left(A*A^{-1}\right)^{-1},$$

which implies $(A*A^{-1})^2\geqslant I$, and hence $A*A^{-1}\geqslant I$.

8. *Estimates for* $\prod_{1}^{k}*A$

In this section we present improved estimates from above or below for the Hadamard product $\prod_1^k*A=A*\cdots*A$ (k times).

THEOREM 19. *If A is positive definite, then*

$$\prod_1^k *A \leqslant \prod_{i=1}^k *A^{p_i} \qquad \textit{whenever} \quad p_i \geqslant 0 \textit{ and } \sum_{i=1}^k p_i = k.$$

In particular, if $k \geqslant 2$,

$$\prod_1^k *A \leqslant A^k * I.$$

Proof. This follows immediately from Theorem 12 with $A_i = A^{p_i}$, because then the A_i's commute and $(\prod_{i=1}^k A_i)^{1/k} = A$. ■

The estimate derived from Theorem 16,

$$\prod_1^k *A \leqslant (A^p * I)^{1/p}(A^q * I)^{1/q}(A * I)^{k-2} \qquad \text{if} \quad p,q \geqslant 1,\ \frac{1}{p} + \frac{1}{q} = 1,$$

is always better than that from Theorem 14:

$$\prod_1^k *A \leqslant \rho_A (A * I)^{k-1}.$$

In fact, $A \leqslant (\rho_A)^{1-1/p} \cdot A^{1/p}$ and $A \leqslant (\rho_A)^{1-1/q} \cdot A^{1/q}$ imply

$$(A^p * I)^{1/p}(A^q * I)^{1/q} \leqslant \rho_A \cdot (A * I).$$

Analogously the estimate derived from Theorem 17

$$\prod_1^k *A \geqslant (A^{-p} * I)^{-1/p}(A^q * I)^{1/q}(A * I)^{k-2}$$

$$\text{if} \quad p \geqslant 1 \geqslant q \geqslant \tfrac{1}{2} \text{ and } \frac{1}{q} - \frac{1}{p} = 1,$$

is always better than the estimate from Theorem 14,

$$\prod_1^k *A \geqslant \delta_A \cdot (A * I)^{k-1}.$$

THEOREM 20. *If A is positive definite, then for $k \geqslant 2$*

$$\prod_1^k *A \geqslant k(A*I)^{k-1}\left\{A^{-1}*\left(\prod_{i=1}^{k-1} *A\right)+(k-1)(A*I)^{k-2}\right\}^{-1}(A*I)^{k-1}.$$

In particular,

$$A*A \geqslant 2(A*I)\left(A^{-1}*A+I\right)^{-1}(A*I).$$

Proof. Let A_i be positive definite $(i=1,2,\dots,k)$, and consider the positive definite matrices of higher order defined by

$$X \equiv \prod_{i=1}^k \otimes A_i, \qquad Y \equiv \sum_{i=1}^k \left(A_1 \otimes \cdots \otimes A_{i-1} \otimes I \otimes A_{i+1} \otimes \cdots \otimes A_k\right)$$

and

$$Z \equiv \sum_{i=1}^k \left(A_1 \otimes \cdots \otimes A_{i-1} \otimes A_i^{-1} \otimes A_{i+1} \otimes \cdots \otimes A_k\right)$$
$$+2\sum_{i<j}\left(A_1 \otimes \cdots \otimes A_{i-1} \otimes I \otimes A_{i+1} \otimes \cdots \otimes A_{j-1} \otimes I \otimes A_{j+1} \otimes \cdots \otimes A_k\right).$$

Then since $Z = YX^{-1}Y$ and hence $X = YZ^{-1}Y$, Corollary 3.1 and Lemma 4 yield

$$\Phi_k(X) \geqslant \Phi_k(Y)\Phi_k(Z)^{-1}\Phi_k(Y).$$

With $A_i = A$ $(i=1,2,\dots,k)$ this inequality leads to the assertion of the theorem. ∎

REMARK. The inequality

$$A*A \geqslant 2(A*I)\left\{A^{-1}*A+I\right\}^{-1}(A*I)$$

was proved by Styan [15] by a different method. The proof of Theorem 20 will show that if A and B are positive definite, then

$$A*B \geqslant \left\{(A+B)*I\right\}\left\{A^{-1}*B+B^{-1}*A+2I\right\}^{-1}\left\{(A+B)*I\right\}.$$

COROLLARY 20.1. *If A is positive definite, then for $k \geqslant 2$*

$$\prod_1^k *A \geqslant \left\{k^{-1}\delta_A^{-1}I+(1-k^{-1})(A*I)^{-1}\right\}^{-1}(A*I)^{k-1}.$$

In particular,

$$A*A \geqslant \left\{(\delta_A I)!(A*I)\right\}(A*I).$$

Proof. This follows from Theorem 20, because by Theorem 15

$$A^{-1}*\left(\prod_1^{k-1} *A\right) \leqslant \delta_A^{-1}\cdot(A*I)^{k-1}. \blacksquare$$

The inequality in Corollary 20.1 is an improvement over the inequality

$$\prod_1^k *A \geqslant \delta_A\cdot(A*I)^{k-1},$$

derived from Theorem 15.

REMARK. Neither of the following inequalities is always valid:

$$\tfrac{1}{2}\left\{\rho_A I+(A*I)\right\}(A*I) \geqslant A*A \geqslant \left\{(\delta_A I)\ \#\ (A*I)\right\}(A*I).$$

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